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B.Sc Part-III

R-integral

Conditions of integrability

The necessary and sufficiente condition of a function in two forms.

Th. (First form) A necessary and sufficient condition for the integrability of a bdd function f is that to every $\epsilon > 0$, there corresponds $\delta > 0$ such that for every partition P of $[a, b]$ with norm $\mu(P) < \delta$

$$U(P, f) - L(P, f) < \epsilon$$

$$\underline{\int_a^b} f\,dx = \int_a^b f\,dx = \overline{\int_a^b} f\,dx.$$

Let ϵ be any positive number. By Darboux's theorem there exists $\delta > 0$ such that for every partition P with norm $\mu(P) < \delta$

$$U(P, f) < \overline{\int_a^b} f\,dx + \frac{1}{2}\epsilon$$

$$= \int_a^b f\,dx + \frac{1}{2}\epsilon \quad (1)$$

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$$L(P,f) > \int_a^b f\,dx - \frac{1}{2}\epsilon = \int_a^b f\,dx - \frac{1}{2}\epsilon \quad -(2)$$

OR

$$-L(P,f) < -\int_a^b f\,dx + \frac{1}{2}\epsilon \quad -(3)$$

from (1) and (3), we get on adding

$$U(P,f) - L(P,f) < \epsilon$$

for every partition P with norm $\mu(P) < \delta$!

The condition is sufficient. Let ϵ be be any positive number. For any partition P with norm $\mu(P) < \delta$ (depending on ϵ), we are given that

$$U(P,f) - L(P,f) < \epsilon$$

Also for any partition P, we know that

$$L(P,f) \leq \int_a^b f\,dx \leq \int_a^b f\,dx \leq U(P,f)$$

$$\Rightarrow \int_a^b f\,dx - \int_a^b f\,dx \leq U(P,f) - L(P,f) < \epsilon$$

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Since ϵ is an arbitary positive number, therefore, we see that a non-negative number is less than